

Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year (تخلفات)		Final Term Exam Date: May 23, 2016 Course: Mathematics 1 – B Duration: 3 hours
• The Exam consists of one page Answer All Questions		• No. of questions: 4 Total Mark: 100
Question 1		
(a) If $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 0 \end{bmatrix}$. Find, if possible : $A + B$, $A.B$, $A.B^t$, $ A $, $ A^t.B $.	10	
(b) If $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$.		
(i) Find the eigenvalues and eigenvectors of A .	8	
(ii) Write Hamilton equation and find A^{-1} .	5	
(iii) Find the eigenvalues of : $f(A) = 2^A$	2	
Question 2		
(a) Using the binomial theorem, expand : $\frac{1}{\sqrt{3x+4}}$	4	
(b) Solve the linear system: $3x + 2y - 3z = 4$, $2x - y = 1$, $x + 3y - 3z = 5$.	5	
(c) Find S , S_{10} from each series: (i) $\sum_{r=1}^n (2r - 1)(1 + r^2)$ (ii) $\sum_{r=1}^n \frac{1}{r^2 + 3r + 2}$	8	
(d) Using the mathematical induction, prove that:		
(i) $1 + 3 + 5 + \dots + (2n - 1) = n^2$ (ii) $1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$	8	
Question 3		
(a) Transform the equation $x^2 - 6xy + 9y^2 + 4x + 8y + 15 = 0$ to new axes through $(-2, 1)$.	6	
(b) A point moves so that its distance from the x - axis is half of its distance from the point $(2, 3)$. Find the equation of its locus.	6	
(c) Show that the equation : $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$ represents two straight lines and find the angle between them.	6	
(d) Find the equation of the circle which passes through the points $(1, 2)$, $(8, 9)$ and cuts the circle $x^2 + y^2 = 25$ at right angle.	7	
Question 4		
(a) Find the vertex , focus, directrix and length of latus rectum the parabola : $y^2 + 4x + 2y - 8 = 0$.	6	
(b) Find the vertices, foci and directrix of the ellipse : $4x^2 + 9y^2 = 144$.	6	
(c) Show that the line $2x + y = 4$ is tangent to the circle : $x^2 + y^2 + 6x - 10y + 29 = 0$.	6	
(d) Trace the curve : $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$.	7	

Answer

Question 1

(a) $A + B = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 3 & 2 \end{bmatrix}$, $A.B$ and $|A|$ are not exist.

$$A.B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 6 \end{bmatrix}.$$

$$A \cdot B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 4 \\ 6 & 3 & 12 \\ 8 & 7 & 4 \end{bmatrix}. \quad |A \cdot B| = 0$$

----- (10 Marks)

(b)(i) $|A - \lambda I| = \begin{vmatrix} -1 - \lambda & 2 \\ 2 & 2 - \lambda \end{vmatrix} = (-1 - \lambda)(2 - \lambda) - 4 = \lambda^2 - \lambda - 6 = 0$

Then, the eigenvalues are: $\lambda_1 = 3$, $\lambda_2 = -2$.

From the equation, $\begin{bmatrix} -1 - \lambda & 2 \\ 2 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

For : $\lambda_1 = 3$, $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then $-4x + 2y = 0$, $2x - y = 0$

Then $2x = y = \text{any number except } 0$

Put $x = 1$, we get $y = 2$ and the

corresponding eigenvector is: $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For : $\lambda_2 = -2$, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then $x + 2y = 0$, $2x + 4y = 0$

Then $x = -2y = \text{any number except } 0$

Put $y = 1$, we get $x = -2$ and the

corresponding eigenvector is: $X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

----- (8 Marks)

(b)(ii) The Hamilton equation is: $A^2 - A - 6I = 0$

Since $|A| = -6$. Then inverse A exists. Multiply the Hamilton equations by A^{-1} .

Then $A - I - 6A^{-1} = 0$.

Then $A^{-1} = \frac{1}{6}(A - I) = \frac{1}{6} \left(\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \frac{1}{6} \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$

----- (5 Marks)

(b)(iii) The eigenvalues of $f(A) = 2^A$ are $f(3) = 2^3 = 8$, $f(-2) = 2^{-2} = \frac{1}{4}$.

----- (2 Marks)

Question 2

(a) $\frac{1}{\sqrt{3x+4}} = \frac{1}{2} \left(1 + \frac{3}{4}x \right)^{-\frac{1}{2}} = \frac{1}{2} \left(1 - \frac{3}{8}x + \frac{27}{8 \times 16}x^2 + \dots \right), \quad \left| \frac{3}{4}x \right| < 1$

----- (4 Marks)

$$(b) G = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 5 \\ 3 & 2 & -3 & 4 \\ 2 & -1 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 5 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 6 & -9 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 5 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & 0 & 2 \end{array} \right]$$

We see that rank A is 2 but rank G is 3. Then, there is no solution.

----- (5 Marks)

$$(c)(i) \text{ Since } u_r = (2r - 1)(1 + r^2) = 2r^3 - r^2 + 2r - 1$$

$$\text{Then } S_n = \frac{1}{2}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) + n(n+1) - n$$

$$\text{Then } S = \infty$$

$$S_0 = (50)(121) - \frac{1}{6}(10)(11)(21) + 100 = 5765$$

----- (4 Marks)

$$(c)(ii) \text{ Since } u_r = \frac{3}{r^2+3r+2} = \frac{1}{r+1} - \frac{1}{r+2}. \text{ Then } S_n = \frac{1}{2} - \frac{1}{n+2}.$$

$$\text{Then } S = \frac{1}{2} \text{ and } S_{10} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

----- (4 Marks)

$$(d)(i) \text{ Prove that: } 1 + 3 + 5 + \dots + 2n - 1 = n^2$$

(1) At $n = 1$, the left hand side (L.H.S) of this relation is 1 and the right hand side (R.H.S) of this relation is 1. Then this relation is true at $n = 1$.

(2) Assume that this relation is true at $n = k$.

$$\text{This means that } 1 + 3 + 5 + \dots + 2k - 1 = k^2$$

(3) We shall prove that this relation is true at $n = k + 1$.

$$\text{Or prove that, } 1 + 3 + 5 + \dots + (2k + 1) = (k + 1)^2$$

From step 2, add $2k + 1$ to both sides, we get

$$1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$$

Then this relation is true for all n .

----- (4 Marks)

$$(d)(ii) \text{ prove that: } 1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$$

(1) At $n = 1$, the left hand side (L.H.S) of this relation is 1 and the right hand side (R.H.S) of this relation is 1. Then this relation is true at $n = 1$.

(2) Assume that this relation is true at $n = k$.

$$\text{This means that } 1 + 2 + 4 + \dots + 2^{k-1} = 2^k - 1$$

(3) We shall prove that this relation is true at $n = k + 1$.

Or prove that, $1 + 2 + 4 + \dots + 2^k = 2^{k+1} - 1$

From step 2, add 2^k to both sides, we get

$$1 + 2 + 4 + \dots + 2^{k-1} + 2^k = 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

Then this relation is true for all n .

----- (4 Marks)

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Question (3)

(a) Transform the equation $x^2 - 6xy + 9y^2 + 4x + 8y + 15 = 0$ to new axes through $(-2, 1)$.

Answer

Put $x = x - 2$ and $y = y + 1$

$$(x - 2)^2 - 6(x - 2)(y + 1) + 9(y + 1)^2 + 4(x - 2) + 8(y + 1) + 15 = 0$$

$$\boxed{x^2 - 6xy + 9y^2 - 6x + 38y + 57 = 0}$$

(b) A point moves so that its distance from the x - axis is half of its distance from the point $(2, 3)$. Find the equation of its locus.

Answer

Let the point is $P(x, y)$ its distance from the x - axis is y and its distance from the point

$(2, 3)$ is $\sqrt{(x - 2)^2 + (y - 3)^2}$ the locus described by

$$y = \frac{1}{2} \sqrt{(x - 2)^2 + (y - 3)^2}$$

$$4y^2 = x^2 - 4x + 4 + y^2 - 6y + 9$$

$$\boxed{3y^2 - x^2 + 4x + 6y - 13 = 0}$$

(c) Show that the equation $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$ represents two straight lines, find the angle between them, and find separately the two straight lines.

Answer

The general form for equation which represents two straight lines is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ by comparison we find}$$

$$a = 12, 2h = 7, b = -10, 2g = 13, 2f = 45, c = -35$$

the discriminant is

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} = \frac{1}{8} \begin{vmatrix} 24 & 7 & 13 \\ 7 & -20 & 45 \\ 13 & 45 & -70 \end{vmatrix} = 0$$

hence the given equation represents to two straight lines

since $12x^2 + 7xy - 10y^2 = (3x - 2y)(4x + 5y)$ then

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = (3x - 2y + c_1)(4x + 5y + c_2)$$

$$= 12x^2 + 7xy - 10y^2 + (3c_2 + 4c_1)x + (5yc_1 - 2c_2)y + c_1c_2 = 0$$

Then $(4c_1 + 3c_2) = 13$, $(5yc_1 - 2c_2) = 45$ and $c_1c_2 = 35$

solve the first two equation we get $c_1 = 7$, $c_2 = -5$

Therefore the required separated equations of the lines are

$3x - 2y + 7 = 0$ and $4x + 5y - 5 = 0$ it is noticed that the two lines

$3x - 2y = 0$ and $3x - 2y + c_1 = 0$ are parallel . Also

$4x + 5y = 0$ and $4x + 5y + c_2 = 0$ are parallel lines.

Hence the angle between the two lines represented by the general equation equal the angle

between the two lines represented by the homogenous equation $ax^2 + 2hxy + by^2 = 0$

$$\tan \theta = \pm \frac{2\sqrt{h^2 - ab}}{a + b} = \pm \frac{2\sqrt{(49/4) + 120}}{2} = 11.5 \text{ and } \theta = 85^\circ 1' 48''$$

(d) Find the equation of the circle which passes through the points (8, 9), (1, 2) and cut the circle $x^2 + y^2 = 25$ at a right angle.

Answer

Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Since the circle cut the circle $x^2 + y^2 = 25$ (2)

Then the line segment of the centers and the two radii form a right angle triangle

$$g^2 + f^2 = r_1^2 + r_2^2$$

$$g^2 + f^2 = 25 + g^2 + f^2 - c$$

$$c = 25$$

$$(8,9) \text{ on the circle then } 8^2 + 9^2 + 16g + 18f + 25 = 0$$

$$(1,2) \text{ on the circle then } 1^2 + 2^2 + 2g + 4f + 25 = 0$$

Solve to find f, g

Question (4)

(a) Find the vertex, focus, directrix and length of the latus rectum of the parabolas

$$y^2 + 4x + 2y - 8 = 0$$

Answer

$$y^2 + 4x + 2y - 8 = 0$$

$$(y^2 + 2y + 1) = -4x + 9$$

$$(y + 1)^2 = -4(x + 2.25) \quad a = -1$$

Since a is negative then the curve opens to the left side

Vertex at $(-2.25, -1)$

Axis of symmetry are $x = -2.25, y = -1$

Focus at $(-2.25 + a, -1) = (-2.25 - 1, -1) = (-3.25, -1)$

Length of the latus rectum $= 4a = 4$

(b) Find the vertices, foci, directrix of the ellipse $4x^2 + 9y^2 = 144$

Answer

$$4x^2 + 9y^2 = 144 \text{ . Divide the equation by 144 , we get } \frac{x^2}{36} + \frac{y^2}{16} = 1$$

$$a^2 = 36, b^2 = 16$$

The center at $(0, 0)$ and the vertices at $(\pm 6, 0), (0, \pm 4)$

Major axis = $2a = 12$ and minor axis = $2b = 8$

$$b^2 = a^2(1 - e^2)$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{36}} = \sqrt{\frac{20}{36}} = \frac{\sqrt{5}}{3}$$

Foci at $(\pm ae, 0) = (\pm 2\sqrt{5}, 0)$

Equation of the directrix is $x = \pm \frac{a}{e} = \pm \frac{3 \times 6}{\sqrt{5}} = \pm \frac{18}{5}\sqrt{5}$

(c) Show that the line $2x + y = 4$ is a tangent to the circle $x^2 + y^2 + 6x - 10y + 29 = 0$.

Answer

The given line is tangent if the perpendicular from the center of the circle equal its radius in length

Center of the circle at $(-3, 5)$ and its radius equal $\sqrt{9 + 25 - 29} = \sqrt{5}$

And the perpendicular from the center to the line is $\left| \frac{2(-3) + 5 - 4}{\sqrt{4 + 1}} \right| = \frac{5}{\sqrt{5}} = \sqrt{5}$ then the line is a tangent.

(d) Trace the curve $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

Answer

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 9 & -12 & -9 \\ -12 & 16 & -101/2 \\ -9 & -101/2 & 19 \end{vmatrix} =$$

$h^2 = (-12)^2 = 144$, $ab = (9)(16) = 144$. Since $h^2 = ab$ The conic section is a parabola

The equation is $(3x - 4y)^2 - 18x - 101y + 19 = 0$ and the straight line $3x - 4y = 0$ as a new axis of x

We rotate the axes by angle θ such that $\tan \theta = \frac{3}{4}$ substitute in the equation by

$$x = X \cos \theta - Y \sin \theta = \frac{4X - 3Y}{5}$$

$$y = X \sin \theta + Y \cos \theta = \frac{3X + 4Y}{5}$$

The equation becomes $25Y^2 - 75X - 70Y + 19 = 0$

and complete the square we get $\left(Y - \frac{7}{5}\right)^2 = 3\left(X + \frac{2}{5}\right)$ (*)

take the point $A\left(-\frac{2}{5}, \frac{7}{5}\right)$ as the origin for the new axes OX, OY , and The focus at $F\left(\frac{7}{20}, \frac{7}{5}\right)$

then the equation of the cone with respect to the new coordinates is $Y^2 = 3X$ vertex at A

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Final Exam and ILOs

Course Title: Mathematics 1-B

Code: EMP 021

Questions	ILOs		
	Knowledge and Understanding	Intellectual Skills	Professional and Practical Skills
	a.1	b.1	c.1
Q1	√	√	√
Q2	√	√	
Q3	√		√
Q4	√	√	√

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